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The effect of a short unheated length and a concentrated heat source on the heat transfer through a turbulent boundary layer

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Abstract—An approximate analytical solution of heat transfer through a two-dimensional turbulent boundary layer with variable heat flux along a flat plate is derived. The effect of a short unheated length on a wall is investigated. In addition, the effect of the combination of a short unheated length and a heated line source midway along the adiabatic strip is evaluated. Numerical solutions are also carried out, utilizing a finite difference marching procedure for parabolic type equations in boundary layers. Results are presented for heat transfer solutions both as a function of distance from the leading edge and as a function of local parameters.

1. INTRODUCTION

CONVECTIVE heat transfer through a turbulent boundary layer is a specific case of the general phenomena of turbulence and turbulent fluid flows. As it is well known, no purely theoretical solution of fluid dynamics for the turbulent boundary layer exists at the present. Consequently there are no exact theoretical solutions available for heat transfer in the turbulent boundary layer, but their occurrence in nature and technology is so frequent that it has been necessary to learn as much as possible about them.

Extensive literature resources concerning constant-property flow with negligible viscous dissipation have been cited in the book by Kays and Crawford [1]. A few technically useful relations can be obtained with some simplified approximations. The energy equation with constant fluid properties is linear in temperature. Hence, evaluation of heat transfer from a flat plate can be decoupled from the momentum equations. Numerous research works have been devoted in the past years to the problems such as heat transfer from a plate with constant wall temperature or constant heat flux. These works provide basic building blocks for constructing solutions for more complicated problems. Computation of heat transfer from a flat plate with an arbitrary wall temperature distribution or wall heat flux distribution can be performed by employing superposition techniques.

In practical engineering problems, non-uniform temperature or wall heat flux distributions across surfaces are encountered much more frequently than the somewhat standard academic-type problems with constant temperature or wall heat flux along a surface. Hence, some simplified approximate solutions for the aforementioned heat transfer problems are highly desirable from a technical point of view.

The major task of this work was to carry out a theoretical analysis of heat transfer through a turbulent boundary layer with variable heat flux along a flat plate. The effect of a short unheated length and a concentrated heat source, which is frequently experienced in many industrial applications, such as the cooling problems of electronic elements, boiling heat transfer, etc., was evaluated. The appropriateness of applying Reynolds' analogy and constant turbulent Prandtl number in the well-established approximate method was checked with the aid of numerical solutions compared with some available experimental data.

The approximate analytical solution of the heat transfer through a turbulent boundary layer with variable heat flux along a flat plate was used as a guidance for a laboratory experiment to evaluate the effect of a short unheated length and a concentrated heat source on a wall. Results are presented for turbulent boundary layers of air on a flat plate. The relations of Stanton number St vs Reynolds numbers, Re_x and Re_{Δ_2}, for three cases are evaluated (Fig. 1).

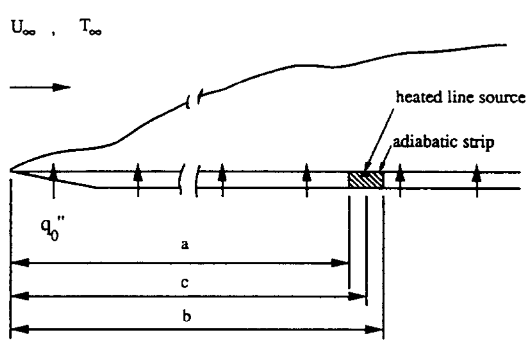

FIG. 1. Sketch of the problem.

NOMENCLATURE

A^+	constant for turbulence model, equation (47)
a	location where the adiabatic wall starts
b	location where the adiabatic wall ends
C	constant
c	location of the concentrated heat source, midway between a and b
C_p	specific heat
f	function for heat transfer coefficient, equation (A1)
g	function for heat transfer coefficient, equation (A2)
h	heat transfer coefficient
I	integrals defined in equations (13), (20) and (21)
k	thermal conductivity
l	mixing length
Pe	Peclet number
Pr	Prandtl number
Q_c	heat flow from the concentrated heat source
q_0	wall heat flux
Re	Reynolds number
St	Stanton number
T	temperature
U	streamwise velocity
V	cross stream velocity
x	streamwise coordinate
y	cross stream coordinate
y_l	characteristic length
y^+	dimensionless distance.

Greek symbols

α	thermal diffusivity
β_1	beta function
β_r	incomplete beta function
Γ	gamma function
Δ_2	enthalpy thickness
δ_{99}	boundary layer thickness
ε	infinitesimal distance
ε_H	eddy diffusivity for heat transfer
ε_M	eddy diffusivity for momentum transfer
η	dimensionless variable
κ	von Karman constant
λ	constant for turbulence model, equation (46)
μ	dynamic viscosity
ν	kinematic viscosity
ξ	local variable
ρ	density
τ_s	shear stress on the wall.

Subscripts and superscripts

a	at location a
b	at location b
c	at location c
t	turbulent
∞	free stream
0	wall
$''$	flux per unit area.

(1) A classical flat-plate, heated, two-dimensional boundary layer. The heat flux is uniform and heating begins at the leading edge.

(2) Same as (1) except there is an adiabatic strip 0.0254 m long, 1.397 m downstream of the leading edge.

(3) Same as (2) except there is a heated line source midway along the adiabatic strip. The energy to the line source is equivalent to the same length of heated surface occupied by the adiabatic strip.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

For a steady, two-dimensional, constant-property turbulent boundary layer over a flat plate with negligible body force and negligible viscous dissipation, the governing equations can be expressed as follows.

Continuity equation

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0. \tag{1}$$

Momentum equation

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left[(\nu+\varepsilon_M)\frac{\partial U}{\partial y}\right]. \tag{2}$$

Energy equation

$$U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}=\frac{\partial}{\partial y}\left[(\alpha+\varepsilon_H)\frac{\partial T}{\partial y}\right] \tag{3}$$

where ε_M and ε_H are the eddy diffusivities for momentum and heat transfer, respectively. These quantities must be determined by a turbulence model, which will be discussed later. It also should be noted that all of the quantities U, V and T are time-averaged mean values.

The inlet conditions at the leading edge and the boundary conditions for velocity fields are the same for all three cases, which can be expressed as follows:

$$\begin{aligned} x=0:&\quad U=U_\infty,\quad V=0,\quad T=T_\infty\\ y=0:&\quad U=0,\quad V=0\\ y\to\infty:&\quad U=U_\infty,\quad V=0. \end{aligned} \tag{4}$$

For the energy equation, boundary conditions at the outer layer are taken as the same, while the conditions on the wall are different

$$y \to \infty: \quad T = T_\infty. \tag{5}$$

(1) Case 1

$$y = 0: \quad q''_0 = -k \cdot \partial T/\partial y = q_0. \tag{6}$$

(2) Case 2

$$y = 0: \quad q''_0 = q_0 \quad \text{for } x < a \text{ and } x > b$$
$$q''_0 = 0 \quad \text{for } a \leqslant x \leqslant b. \tag{7}$$

(3) Case 3

$$y = 0: \quad q''_0 = q_0 \quad \text{for } x < a \text{ and } x > b$$
$$q''_0 = 0 \quad \text{for } a \leqslant x < c \text{ and } c < x \leqslant b$$
$$q''_0 = q_c \quad \text{for } x = c$$

where

$$\int_{c-\varepsilon}^{c+\varepsilon} q_c \, dx = q_0(b-a). \tag{8}$$

3. APPROXIMATE ANALYTICAL SOLUTION

As mentioned before, analytical solutions can be developed by accepting some degree of approximation in return for a simplified usable result. The energy integral equation was used, following the same general procedure of a laminar boundary layer.

3.1. *Heat transfer solution as a function of distance from the leading edge*

The solution of the turbulent boundary layer with constant free stream velocity along a semi-infinite plate with unheated starting length is treated first. The details can be found in ref. [1]. A basic assumption is that the velocity and temperature profiles can be expressed by a 1/7 power law. Reynolds' analogy is also applied, i.e. $Pr = 1$ and $Pr_t = 1$, so is the fact that the boundary layer thickness varies with a 4/5 power of length. Further assuming the starting-length correction for a step-wall temperature independent of Prandtl number leads to a simple relation as follows:

$$St_x \cdot Pr^{0.4} = 0.0287 Re_x^{-0.2} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-1/9}. \tag{9}$$

Following the same procedure for the laminar boundary layer, the heat transfer solution can be modified to yield the wall temperature as a function of heat input [2, 3]. The result is given in the following form:

$$T_0(x) - T_\infty = \int_0^x q''_0(\xi) g(\xi, x) \, d\xi \tag{10}$$

where $q''_0(\xi)$ is the arbitrary prescribed surface heat flux and

$$g(\xi, x) = \frac{3.42}{k} Pr^{-0.6} Re_x^{-0.8} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9}. \tag{11}$$

Then equation (10) becomes

$$T_0(x) - T_\infty = \frac{3.42}{k} Pr^{-0.6} Re_x^{-0.8} \int_0^x q''_0(\xi) \times \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\xi. \tag{12}$$

3.1.1. *Case* 1. The simplest example for a specified heat flux is a constant value. With $q''_0(\xi)$ a constant, equation (12) can be expressed in terms of a beta function.

Define an integral

$$I \equiv \int_0^x \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\xi = \frac{10x}{9} \beta_1\left(\frac{1}{9}, \frac{10}{9}\right) \tag{13}$$

then the heat transfer solution can be presented via temperature as

$$T_0(x) - T_\infty = \frac{33.6}{k} Pr^{-0.6} Re_x^{-0.8} q_0 x \tag{14}$$

or in terms of Stanton number

$$St = 0.02976 Pr^{-0.4} Re_x^{-0.2}. \tag{15}$$

3.1.2. *Case* 2. A step-function of heat flux was prescribed to the wall. Superposition of heat flux was applied in equation (12).

For $x < a$, the same result as case 1 was obtained.

For $a \leqslant x \leqslant b$

$$T_0(x) - T_\infty = q_0 \int_0^a g(\xi, x) \, d\xi. \tag{16}$$

For $x > b$

$$T_0(x) - T_\infty = q_0 \left[\int_0^a g(\xi, x) \, d\xi + \int_0^x g(\xi, x) \, d\xi - \int_0^b g(\xi, x) \, d\xi\right]. \tag{17}$$

The integrals in equations (16) and (17) can be transformed into a so-called incomplete beta function defined as

$$\beta_r(m, n) \equiv \int_0^r z^m (1-z)^{n-1} \, dz = \beta_1(m, n) - \beta_{1-r}(m, n). \tag{18}$$

The incomplete beta function is shown in Fig. 2.

Define two more variables

$$\eta_a(x) \equiv 1 - \left(\frac{a}{x}\right)^{9/10}$$
$$\eta_b(x) \equiv 1 - \left(\frac{b}{x}\right)^{9/10} \tag{19}$$

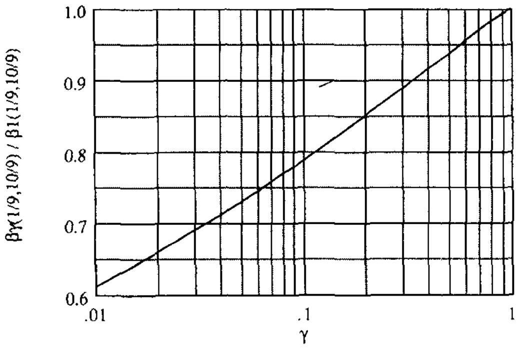

FIG. 2. Normalized incomplete beta function with respect to $\beta_1(1/9, 10/9) = 8.8439$.

then the integral in equation (16) becomes

$$I_a \equiv \int_0^a \left[1-\left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\xi = \frac{10x}{9}\left[\beta_1\left(\frac{1}{9},\frac{10}{9}\right) - \beta_{\eta_a}\left(\frac{1}{9},\frac{10}{9}\right)\right]. \quad (20)$$

Similarly

$$I_b \equiv \int_0^b \left[1-\left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\xi = \frac{10x}{9}\left[\beta_1\left(\frac{1}{9},\frac{10}{9}\right) - \beta_{\eta_b}\left(\frac{1}{9},\frac{10}{9}\right)\right]. \quad (21)$$

In summary, the solutions for case 2 can be listed as follows:

(1) $x < a$, same as case 1.
(2) $a \leqslant x \leqslant b$

$$T_0(x)-T_\infty = \frac{3.42}{k} Pr^{-0.6}\, Re_x^{-0.8}\, q_0 I_a. \quad (22)$$

(3) $x \geqslant b$

$$T_0(x)-T_\infty = \frac{3.42}{k} Pr^{-0.6}\, Re_x^{-0.8}\, q_0 (I_a + I - I_b) \quad (23)$$

or

$$St = \frac{x\, Pr^{-0.4}\, Re_x^{-0.2}}{3.42(I_a + I - I_b)}. \quad (24)$$

3.1.3. *Case* 3. A line source is placed at $x = c$, the midpoint of a and b, which can be treated as a delta function with a certain heat input

$$\int_{c-\varepsilon}^{c+\varepsilon} q_c\, d\xi = q_0(b-a) \equiv Q_c. \quad (25)$$

The line source is a limiting case of equation (10), for which the interval approaches zero and the source intensity increases to keep the heat input a finite value. Therefore

$$T_0(x)-T_\infty = \Sigma_n g(\xi_n, x) Q(\xi_n)$$
$$\text{for } \xi_n > x, g(\xi_n, x) = 0. \quad (26)$$

For $c \leqslant x \leqslant b$

$$T_0(x)-T_\infty = q_0 \int_0^a g(\xi, x)\, d\xi + g(c, x) Q_c. \quad (27)$$

For $x > b$

$$T_0(x)-T_\infty = q_0 \int_0^a g(\xi, x)\, d\xi + g(c, x) Q_c + q_0 \int_0^x g(\xi, x)\, d\xi - q_0 \int_0^b g(\xi, x)\, d\xi. \quad (28)$$

Hence, the solutions for case 3 are:

(1) $x < c$, same as case 2.
(2) $c \leqslant x \leqslant b$

$$T_0(x)-T_\infty = \frac{3.42}{k} Pr^{-0.6}\, Re_x^{-0.8}\, q_0 \left\{ I_a + (b-a) \times \left[1-\left(\frac{c}{x}\right)^{9/10}\right]^{-8/9} \right\}. \quad (29)$$

(3) $x \geqslant b$

$$T_0(x)-T_\infty = \frac{3.42}{k} Pr^{-0.6}\, Re_x^{-0.8}\, q_0 \times \left\{ I_a + I - I_b + (b-a)\left[1-\left(\frac{c}{x}\right)^{9/10}\right]^{-8/9} \right\} \quad (30)$$

or

$$St = \frac{x\, Pr^{-0.4}\, Re_x^{-0.2}}{3.42\left\{ I_a + I - I_b + (b-a)\left[1-\left(\frac{c}{x}\right)^{9/10}\right]^{-8/9} \right\}}. \quad (31)$$

3.2. *Heat transfer solution as a function of local parameters*

In some situations, the relation between heat transfer coefficient and local parameters, say Re_{Δ_2}, is more desirable. The energy integral equation as shown below was used to derive this kind of relation [1]

$$\frac{q_0''}{C_p} = \frac{1}{R}\frac{d}{dx}\{\Delta_2 R \rho_\infty U_\infty [T_0(x) - T_\infty]\}. \quad (32)$$

For a constant-property flow over a flat plate with constant free stream velocity, the equation can be simplified to

$$\Delta_2 = \frac{\int_0^x q_0''(\xi)\, d\xi}{\rho_\infty U_\infty C_p [T_0(x) - T_\infty]}. \quad (33)$$

3.2.1. *Case* 1. For constant heat flux, a simple relation between Δ_2 and St can be obtained

$$\Delta_2 = St \cdot x. \tag{34}$$

The Stanton number generally can be expressed in the form

$$St = C \cdot Re_x^{-n} \tag{35}$$

then the relation between Δ_2 and x for case 1 can be derived as follows:

$$\Delta_2 = C\, Re_x^{-n} \cdot x = C \frac{v^n}{U_\infty^n} x^{1-n} \tag{36}$$

or

$$x = \left(\frac{\Delta_2}{C} \frac{U_\infty^n}{v^n}\right)^{1/(1-n)}. \tag{37}$$

Substituting into equation (35), one obtains

$$St = C^{1/(1-n)} \cdot Re_{\Delta_2}^{n/(n-1)}. \tag{38}$$

The constants in equation (35) can be determined by comparing equation (35) with equation (15), i.e. $C = 0.02976 Pr^{-0.4}$ and $n = 0.2$. Then the relation between St and Re_{Δ_2} can be obtained from equation (37)

$$St = 0.01236 Pr^{-1/2} Re_{\Delta_2}^{-1/4}. \tag{39}$$

It should be noted that there are no such simple relations as equation (39) for cases 2 and 3. However, the relation between Δ_2 and x can be evaluated through equation (33) since the temperature difference $T_0(x) - T_\infty$ has been calculated in Section 3.1. Thus the relation between St and Re_{Δ_2} can then be obtained.

In the following, the relation between Δ_2 and x are stated for cases 2 and 3.

3.2.2. *Case* 2.

(1) $x < a$, same as case 1.
(2) $a \leqslant x \leqslant b$

$$\Delta_2 = \frac{q_0 a}{\rho_\infty U_\infty C_p [T_0(x) - T_\infty]}. \tag{40}$$

(3) $x > b$

$$\Delta_2 = \frac{q_0 (a + x - b)}{\rho_\infty U_\infty C_p [T_0(x) - T_\infty]}. \tag{41}$$

3.2.3. *Case* 3.

(1) $x < c$, same as case 2.
(2) $c \leqslant x \leqslant b$

$$\Delta_2 = \frac{q_0 b}{\rho_\infty U_\infty C_p [T_0(x) - T_\infty]}. \tag{42}$$

(3) $x > b$

$$\Delta_2 = \frac{q_0 x}{\rho_\infty U_\infty C_p [T_0(x) - T_\infty]}. \tag{43}$$

4. NUMERICAL SOLUTION

In order to solve the governing equations with the prescribed boundary conditions, a turbulence model has to be introduced to evaluate the eddy quantities ε_M and ε_H, or in turn, the eddy viscosity μ_t and the turbulent Prandtl number Pr_t.

The theory of turbulent wall shear layers is presently in a state of intense study, and new breakthroughs are continually in sight. But the simplest of all the schemes proposed remains the very old Prandtl mixing-length model, and with new information available on the very important behaviour of the viscous sublayer, the mixing-length model provides a remarkably adequate basis for many engineering applications, especially for some simple flow patterns. The following calculations were based on this turbulence model:

$$\mu_t = \rho l^2 |\partial U / \partial y|. \tag{44}$$

To evaluate the mixing length l, the outer region of the boundary layer and the near wall region must be considered separately. For flows remote from walls, l is usually taken as uniform across the layer and proportional to the thickness of the layer. For a boundary layer on a wall, the variation of l in the outer part is similar to that in free turbulent flows, but l is proportional to the distance from the wall for the near wall region. Escudier's formula is usually recommended by investigators working on this field [4, 5]

$$l = \kappa y \quad \text{for } 0 < y < \lambda y_l / \kappa \tag{45}$$

$$l = \lambda y_l \quad \text{for } y \geqslant \lambda y_l / \kappa \tag{46}$$

where y is the distance from the wall, λ and κ are constants; κ is usually called the von Karman constant and y_l is a characteristic thickness of the layer, usually taken as δ_{99} (the boundary layer thickness at the point where $U/U_\infty = 0.99$).

However, for a region very close to the wall, called the viscous sublayer, equation (45) needs to be modified. Then, van Driest's hypothesis is applied at the near wall region, which is

$$l = \kappa y [1 - \exp(-y^+ / A^+)] \tag{47}$$

where A^+ is a constant and y^+ a dimensionless distance defined as

$$y^+ \equiv y (\rho \tau_s)^{1/2} / \mu$$

in which τ_s is the shear stress on the wall.

The turbulent Prandtl number must be specified for applying various turbulent transport theories to heat transfer. Most workers have solved the energy equation by assuming a constant Pr_t. A marching procedure for solving parabolic equations of boundary layers described in ref. [4] was utilized for the present work. The empirical constants of the Prandtl mixing-length hypothesis were adopted from ref. [1]: i.e. $\lambda = 0.085$, $\kappa = 0.41$ and $A^+ = 25$. A value of 0.9 for the turbulent Prandtl number was applied for the constant Pr_t results.

5. RESULTS AND DISCUSSIONS

Results were presented for turbulent boundary layers of air on a flat plate. Heat transfer solutions were obtained with free stream temperature, $T_\infty = 20°C$, and free stream velocity, $U_\infty = 15$ m s^{-1}. Boundary conditions on the wall were prescribed heat flux, with $q_0 = 240$ W m^{-2}, $a = 1.397$ m, $b = 1.4224$ m and $Q_c = q_0(b-a)$. Fluid properties, which were obtained from ref. [1], were regarded as constants based on the free stream temperature: $\rho = 1.2047$ kg m^{-3}, $\mu = 1.817 \times 10^{-5}$ kg m^{-1} s^{-1}, $C_p = 1004$ J kg^{-1} K^{-1}, $k = 2.563 \times 10^{-2}$ W m^{-1} K^{-1} and $Pr = 0.712$. The relation between Re_x and x can be evaluated as $Re_x = 9.945 \times 10^5 x$ or $x = 1.0055 \times 10^{-6} Re_x$. Location $x = a$ corresponds to $Re_a = 1.389 \times 10^6$ and $x = b$ corresponds to $Re_b = 1.415 \times 10^6$.

5.1. *Heat transfer solutions as functions of x (or Re_x)*

For uniform heat flux, it was obtained from equation (15) that

$$CTX \equiv St\, Pr^{0.4}\, Re_x^{0.2} = 0.02976.$$

The numerical solution for case 1 with constant Pr_t showed that the above constant varied within 0.03007 and 0.02967 for the range of $5 \times 10^5 < Re_x < 5 \times 10^6$, which gave the deviation between analytical and numerical solutions within 1%.

A comparison of temperature difference distribution for a uniform heat source and for a discontinuous heat source dissipating the same power has been presented in ref. [2]. The temperature difference distribution for a plate with pulse heat input was also presented in ref. [3]. It was found that there is a steep change (decrease) of temperature distribution in the downstream vicinity of the starting point of unheated walls, and similarly, there is a steep increase of temperature distribution in the immediate downstream region of the end point of unheated walls. The situation in the present work is somewhat similar but not equivalent to the above cases. However, the trend of the variation of temperature distribution for an unheated wall can be applied to the present situation. Since the temperature difference, $T_0 - T_\infty$, in the unheated region is lower than that of uniform heat flux, this situation is still sustained in the immediate downstream region of the end point of the unheated wall. Thus, it is expected that St in this region for the case with an unheated wall is higher than that for uniform heat flux. However, the effect of an unheated wall on St decays rapidly downstream since there is a sharp increase of temperature distribution, and this effect is only restricted in the vicinity of the location where heat flux changes.

Figure 3 shows the relation of Stanton number vs Re_x. It can be seen that a short unheated length on a wall ($a \leqslant x \leqslant b$) has a strong effect on St in the immediate vicinity downstream of the unheated length. For example, at $x = b+\varepsilon$ (ε is an infinitesimal distance), $St_1 = 2.007 \times 10^{-3}$, $St_2 = 5.6 \times 10^{-3}$ and $St_3 = 4.09 \times 10^{-3}$, where St_1, St_2 and St_3 are the Stanton numbers of cases 1, 2 and 3, respectively. As expected, variations of St for cases 2 and 3 decay rapidly downstream. For example, at the location $Re_x = 1.5 \times 10^6$ (i.e. $x = 1.5083$ m, 0.0859 m downstream from b), $St_2/St_1 = 1.021$ and $St_3/St_1 = 1.011$. The variation of St at this location is reduced to around 2% for case 2 and around 1% for case 3 as compared to that of case 1. Hence, it can be concluded that the effect of an unheated wall on St is substantial in the neighbourhood of the location where heat flux changes, although this effect decays rapidly downstream.

Additional details were examined in the neighbourhood region of the unheated wall. Figure 4 presents the temperature difference between the wall and free stream, $T_0 - T_\infty$, vs the distance from the leading edge, x. Figure 5 shows the relation between St and Re_x. Similar trends of temperature distribution were observed as those presented in ref. [2]. The effect of the line source applied in case 3 can be clearly seen from these figures. The line source at location c, midpoint between a and b, was considered as an infinite heat flux with infinitesimal width the energy input of which was equivalent to a heat source with the same length of unheated wall (i.e. $Q_c = q_0(b-a)$). Under this condition, the temperature difference of case 3 approached infinity at location c; however, it decayed even faster than that of case 2. Finally, this yielded a result that $T_0 - T_\infty$ of case 3 lays between those of cases 1 and 2 for the downstream region from b. Thus St_3 lays between St_1 and St_2 in the downstream region of the unheated wall. This is to say, the line source applied in case 3 can somewhat but not totally compensate the effect of an unheated wall on Stanton number, even though the energy input of the line source is equivalent to the energy deficit of case 2. The numerical solutions for cases 1 and 2 are shown in Fig. 6. In general, the same trend as the analytical counterparts is predicted.

5.2. *Heat transfer solutions as functions of local parameters*

A simple relation for the uniform heat flux case has been derived before, equation (39), which gives

$$CTD2 \equiv St\, Pr^{1/2}\, Re_{\Delta_2}^{1/4} = 0.01236.$$

The numerical solution of case 1 showed that $CTD2$ varied between 0.01212 and 0.01225 for the same region as before, the deviation of which was within 2% compared to the analytical solution.

Figure 7 shows the relation between enthalpy thickness Δ_2 and x. For the uniform heat flux case, Δ_2 increases monotonously with x. However, the unheated wall and line source have strong effects on Δ_2 in the vicinity where the heat flux variation occurs.

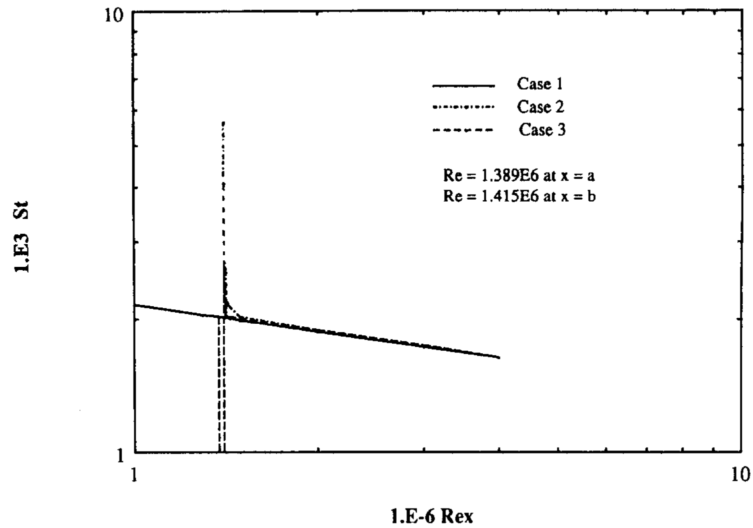

FIG. 3. Analytical solutions for the relation of St vs Re_x.

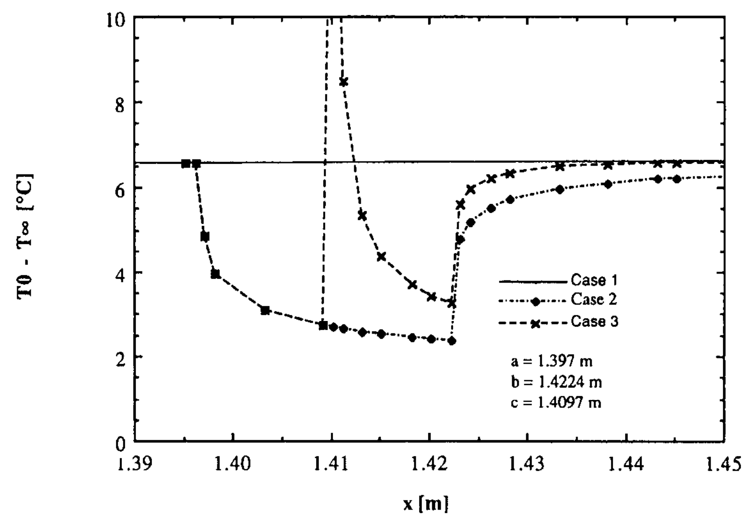

FIG. 4. Analytical solutions for the relation of $(T_0 - T_\infty)$ vs x at the neighbourhood of the unheated wall.

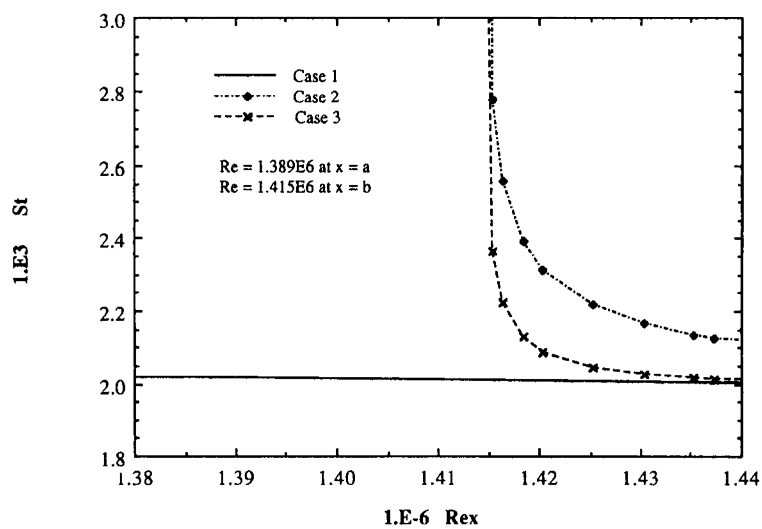

FIG. 5. Analytical solutions for the relation of St vs Re_x at the neighbourhood of the unheated wall.

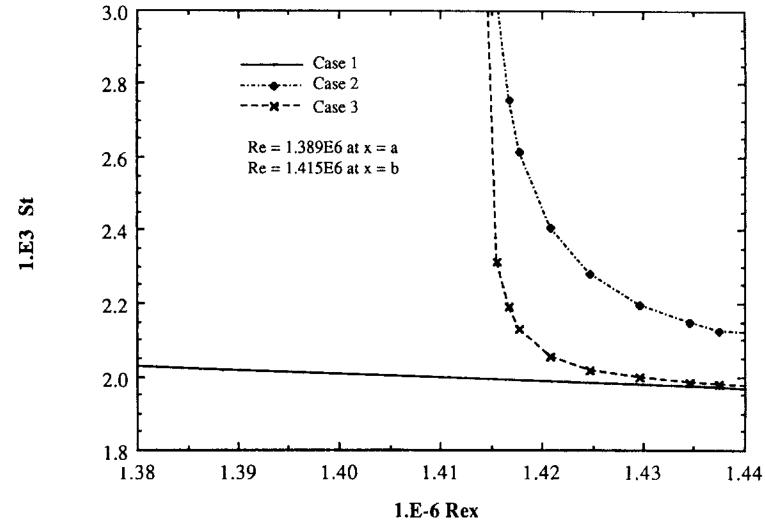

FIG. 6. Numerical solutions for the relation of St vs Re_x at the neighbourhood of the unheated wall.

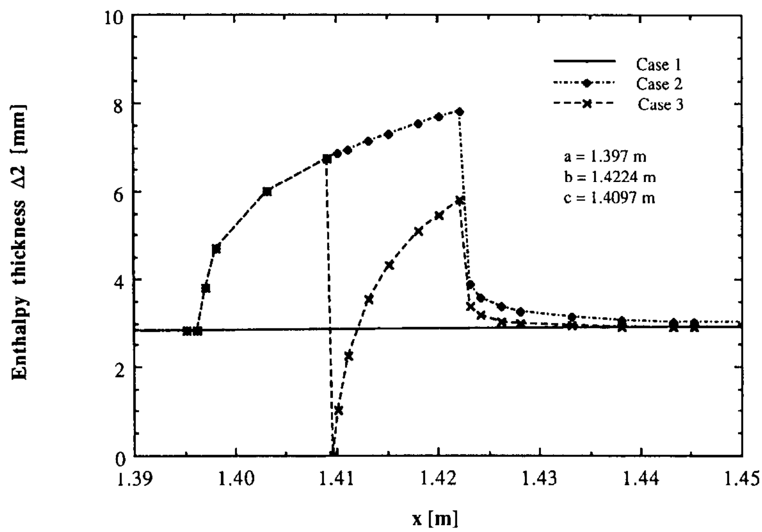

FIG. 7. Analytical solutions for the relation of Δ_2 vs x at the neighbourhood of the unheated wall.

It is observed that Δ_2 increases sharply where heat flux drops to zero and decreases rapidly where some certain amount of heat flux is applied (Δ_2 drops to zero for a line source which corresponds to an infinite temperature difference). However, Δ_2 then recovers its monotonous increase with x at a distance further downstream of the unheated segment.

In the neighbourhood of the unheated wall Δ_2 is not a monotonous function of x and thus it is inappropriate to use Δ_2 as an independent parameter for interpretation of heat transfer performance. Hence, the relations of St vs Re_{Δ_2} for cases 2 and 3 were presented only for the downstream region of the unheated wall where Δ_2 is a monotonous function of x. It is seen in Fig. 8 that variations of St for the three cases were within 5%. Finally, numerical solutions for non-uniform heat flux cases are shown in Fig. 9. In general, the same trend as the counterparts of analytical solution is observed.

6. CONCLUSIONS

Heat transfer to the wall of a turbulent boundary layer has been studied, both from an analytical consideration and by a numerical method. The effect of a short unheated length and a concentrated heat source on a wall has been evaluated. Some concluding remarks can be drawn as follows.

(1) A short unheated wall has a strong effect on the heat transfer solution in the vicinity immediately downstream of the end point of the unheated wall.

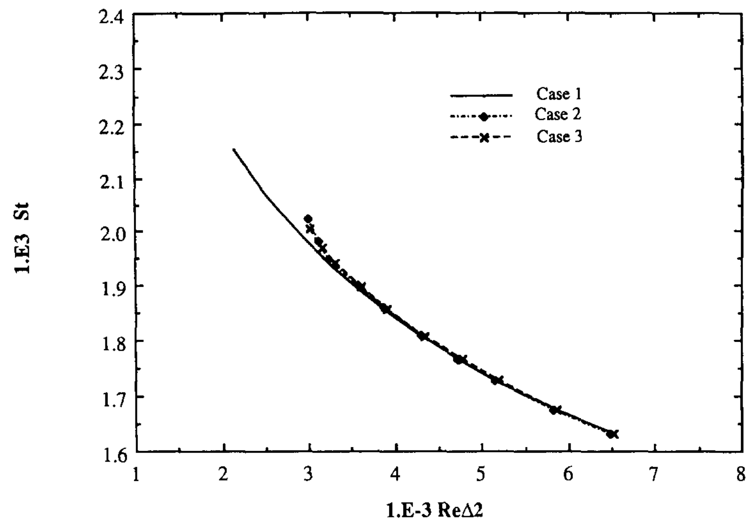

FIG. 8. Analytical solutions for the relation of St vs Re_{Δ_2} downstream of the unheated wall.

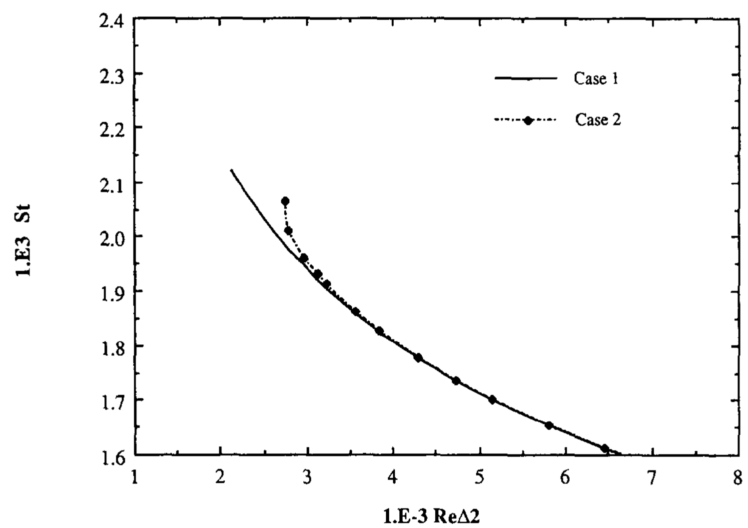

FIG. 9. Numerical solutions for the relation of St vs Re_{Δ_2} downstream of the unheated wall.

However, this effect decays rapidly and becomes negligible further downstream.

(2) The line source applied in case 3 can somewhat but not totally compensate for the effect on Stanton number caused by an unheated wall, although the energy input of the line source is the same as the energy deficit due to the adiabatic strip in case 2.

(3) Comparison of the calculated results showed that the analytical solution based on Reynolds' analogy and some simplified approximations provides a simple but reasonable analysing tool. This approximation is technically useful. Comparison of calculated results with available experimental data for the uniform heat flux cases given in Appendix B showed satisfactory agreement.

REFERENCES

1. W. M. Kays and M. E. Crawford, *Convective Heat and Mass Transfer*, 2nd Edn. McGraw-Hill, New York (1980).
2. J. Klein and M. Tribus, Forced convection from nonisothermal surfaces, ASME Paper 53-SA-46, ASME Semiannual Meeting (1953).
3. W. C. Reynolds, W. M. Kays and S. J. Kline, Heat transfer in the turbulent incompressible boundary layer. III: Arbitrary wall temperature and heat flux, NASA Memo. No. 12-3-58W, Washington, DC (1958).
4. S. V. Patankar and D. B. Spalding, *Heat and Mass Transfer in Boundary Layer*. Morgan–Grampian, London (1967).
5. B. E. Launder and D. B. Spalding, *Lectures in Mathematical Models of Turbulences*. Academic Press, New York (1972).
6. A. T. Wassel and I. Catton, Calculation of turbulent

boundary layers over flat plates with different phenomenological theories of turbulence and variable turbulent Prandtl number, *Int. J. Heat Mass Transfer* **16**, 1547–1563 (1973).
7. R. L. Simpson, D. G. Whitten and R. J. Moffat, An experimental study of the turbulent Prandtl number of air with injection and suction, *Int. J. Heat Mass Transfer* **13**, 125–143 (1970).
8. R. A. Seban and D. L. Doughty, Heat transfer to turbulent boundary layers with variable free-stream velocity, *Trans. ASME* **78**, 217–223 (1956).

APPENDIX A. PROCEDURE OF THE ANALYTICAL SOLUTION

For certain special cases, the heat transfer coefficient from a step-function solution can be put in the form

$$h(\xi, x) = f(x)(x^{\gamma} - \xi^{\gamma})^{-\alpha}. \quad (A1)$$

Defining a modified form as

$$g(\xi, x) \equiv \frac{(x^{\gamma} - \xi^{\gamma})^{\alpha-1}\gamma\xi^{\gamma-1}}{f(\xi)(-\alpha)!(\alpha-1)!} \quad (A2)$$

then the result is in the following form:

$$T_0(x) - T_\infty = \int_0^x q_0''(\xi) g(\xi, x)\, d\xi \quad (A3)$$

where $q_0''(\xi)$ is the arbitrary prescribed surface heat flux.

From equation (9), the function $f(x)$ and the exponents α and γ can be derived via the relation between the Stanton number St and the heat transfer coefficient h

$$h(\xi, x) = 0.0287 k x^{-0.1} Pr^{0.6} \left(\frac{U_\infty}{\nu}\right)^{0.8} [x^{9/10} - \xi^{9/10}]^{-1/9} \quad (A4)$$

then

$$f(x) = 0.0287 k x^{-0.1} Pr^{0.6} \left(\frac{U_\infty}{\nu}\right)^{0.8}$$

$$\gamma = 9/10, \quad \alpha = 1/9. \quad (A5)$$

Substituting into equation (A2), one obtains

$$g(\xi, x) = \frac{0.9 Pr^{-0.6} Re_x^{-0.8}}{\Gamma(\frac{8}{9})\Gamma(\frac{1}{9}) \cdot 0.0287 k} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} \quad (A6)$$

in which the factorial is expressed as a gamma function through the relation $\Gamma(x+1) = x!$.

In general, numerical tables of the gamma function $\Gamma(x)$ can be found for $1 \leqslant x \leqslant 2$, while the value for the other range can be calculated through the relation $\Gamma(x+1) = x \cdot \Gamma(x)$. Inserting the numerical value of the gamma function into equation (A6), one obtains

$$g(\xi, x) = \frac{3.42}{k} Pr^{-0.6} Re_x^{-0.8} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9}. \quad (A7)$$

Then equation (A3) becomes

$$T_0(x) - T_\infty = \frac{3.42}{k} Pr^{-0.6} Re_x^{-0.8} \int_0^x q_0''(\xi) \times \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\xi. \quad (A8)$$

The simplest example for a specified heat flux is a constant value. With $q_0''(\xi)$ a constant, equation (A8) becomes

$$T_0(x) - T_\infty = \frac{3.42}{k} Pr^{-0.6} Re_x^{-0.8} q_0 \times \int_0^x \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\xi. \quad (A9)$$

Introducing a change of variable

$$\eta \equiv 1 - \left(\frac{\xi}{x}\right)^{9/10} \quad (A10)$$

the integral can be transformed into the form of the beta function which is defined as

$$\beta_1(m, n) \equiv \int_0^1 z^{m-1}(1-z)^{n-1}\, dz \quad (A11)$$

where

$$\beta_1(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)} \quad \text{for } m > 0, n < \infty. \quad (A12)$$

Hence

$$I \equiv \int_0^x \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\xi = \frac{10x}{9} \int_0^1 \eta^{-8/9}(1-\eta)^{1/9}\, d\eta. \quad (A13)$$

Comparing equation (A13) with equation (A11), one finds $m = 1/9$ and $n = 10/9$. Thus

$$I = \frac{10x}{9} \beta_1\left(\frac{1}{9}, \frac{10}{9}\right) = 9.827x. \quad (A14)$$

APPENDIX B. THE TURBULENT PRANDTL NUMBER

As mentioned before, we must know the eddy viscosity and the turbulent Prandtl number in order to solve heat transfer problems. A number of experimental and theoretical investigations have been devoted to obtaining the eddy viscosity. However, since much less is known about the turbulent transport phenomenon like turbulent heat transfer compared to momentum transfer, fewer studies have been made for the turbulent Prandtl number.

The simplest approach is to make a plausible assumption, which reached the conclusion that $Pr_t = 1$, the well-known Reynolds' analogy. This statement is based on a heuristic argument that both momentum and heat are transferred as a result of the motion of eddies in a fully turbulent field. Experimental evidence suggested that Pr_t, at least for air, is of the order of unity over most of the boundary layer except in the region very close to the wall.

In the following, two prediction models for Pr_t are adopted. The first model to be used with the numerical solution in order to take into account the variable Pr_t effect was adopted from ref. [1] and assumed the form of

$$Pr_t = \left\{\frac{1}{2Pr_{t\infty}} + \frac{C_h Pe_t}{\sqrt{(Pr_{t\infty})}} - (C_h Pe_t)^2 \times \left[1 - \exp\left(\frac{-1}{C_h Pe_t \sqrt{(Pr_{t\infty})}}\right)\right]\right\}^{-1} \quad (B1)$$

where $Pe_t = (\varepsilon_M/\nu)Pr$; $Pr_{t\infty} = 0.86$ (value of Pr_t far from the wall, an experimental constant); $C_h = 0.2$ (an experimental constant).

The second model was proposed by Wassel and Catton [6], which contained four adjustable constants chosen to correlate with the data of Simpson *et al.* [7]

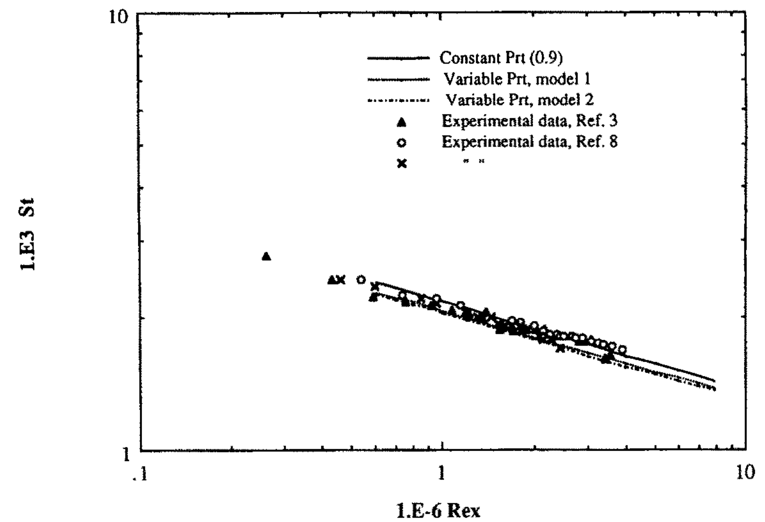

FIG. B1. Comparison of computational solutions with experimental data for uniform heat flux situation (case 1).

$$Pr_t = \frac{C_3}{C_1 Pr} \frac{\left\{1-\exp\left[\frac{-C_4}{(\varepsilon_M/\nu)}\right]\right\}}{\left\{1-\exp\left[\frac{-C_2}{Pr(\varepsilon_M/\nu)}\right]\right\}} \tag{B2}$$

where $C_1 = 0.21$, $C_2 = 5.25$, $C_3 = 0.20$ and $C_4 = 5$. With these constants the predicted values of Pr_t fell within the uncertainty envelope of the experimental results.

For variable Pr_t, the numerical solutions adopting both models described above showed that the values of CTX changed only slightly, which revealed the similar heat transfer behaviour as the constant Pr_t case. For the first model, equation (B1), CTX varied within 0.02843 and 0.02860, which was about a 4.5% deviation. For the second model, equation (B2), CTX varied between 0.02805 and 0.02823, which gave about a 5.8% deviation.

Comparison of experimental data [3, 8] and numerical solutions for both constant and variable Pr_t are presented in Fig. B1. The approximate analytical solution can be regarded as almost identical to the numerical solution for the constant Pr_t case. For experimental data, several different free stream velocities were applied, but the conventional representation of heat transfer results, i.e. St vs Re_x, fell almost together one by another. In general, good agreement between experimental data and calculated results was achieved. It is found that consideration of variable Pr_t provides a better prediction for the heat transfer to the wall for lower Reynolds numbers, $Re_x < 1.5 \times 10^6$; while the constant Pr_t solution seems to match the experiments better for higher Reynolds numbers, $Re_x > 2 \times 10^6$.

EFFET D'UNE COURTE LONGUEUR NON CHAUFFEE ET D'UNE SOURCE CONCENTREE SUR LE TRANSFERT THERMIQUE A TRAVERS UNE COUCHE LIMITE TURBULENTE

Résumé—On établit une solution analytique approchée du transfert thermique à travers une couche limite turbulente bidimensionnelle avec flux thermique variable le long d'une plaque plane. L'effet d'une petite longueur non chauffée sur la paroi est étudié. En outre on évalue l'effet de la combinaison d'une zone courte non chauffée et d'une source linéaire au milieu de la bande adiabatique. Des solutions numériques sont données qui utilisent une technique aux différences finies pour les équations paraboliques de couche limite. Des résultats sont présentés qui donnent le transfert thermique à la fois en fonction de la distance au bord d'attaque et des paramètres locaux.

EINFLUSS EINER KURZEN UNBEHEIZTEN STRECKE UND EINER KONZENTRIERTEN WÄRMEQUELLE AUF DEN WÄRMETRANSPORT DURCH EINE TURBULENTE GRENZSCHICHT

Zusammenfassung—Es wird eine analytische Näherungslösung für den Wärmetransport durch eine zweidimensionale turbulente Grenzschicht entlang einer ebenen, unterschiedlich beheizten Platte hergeleitet. Der Einfluß einer kurzen unbeheizten Strecke an der Wand wird untersucht. Zusätzlich wird der kombinierte Einfluß einer kurzen unbeheizten Strecke und einer linienförmigen Wärmequelle in der Mitte des adiabaten Streifens untersucht. Die parabolischen Grenzschichtgleichungen werden mit Hilfe eines Finite-Differenzen-Verfahrens numerisch gelöst. Die Ergebnisse für den Wärmeübergang werden als Funktion der Lauflänge und als Funktion örtlicher Parameter dargestellt.

ВЛИЯНИЕ КОРОТКОГО НЕНАГРЕТОГО УЧАСТКА И СОСРЕДОТОЧЕННОГО ИСТОЧНИКА ТЕПЛА НА ТЕПЛОПЕРЕНОС ЧЕРЕЗ ТУРБУЛЕНТНЫЙ ПОГРАНИЧНЫЙ СЛОЙ

Аннотация—Приводится приближенное аналитическое решение задачи теплопереноса через двумерный турбулентный пограничный слой с переменным тепловым потоком вдоль плоской пластины. Исследуется влияние ненагретого участка малой длины на стенке. Оценивается также эффект комбинации короткого ненагретого участка и линейного продольного источника тепла в середине адиабатической пластинки. Описываются численные решения, полученные с использованием конечно-разностного метода прогонки для уравнений параболического типа в пограничных слоях. Представлены результаты решения задач теплопереноса в зависимости как от расстояния от передней кромки, так и от локальных параметров.